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HOMOLOGY AND SOME HOMOTOPY DECOMPOSITIONS FOR THE JAMES FILTRATION ON SPHERES

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ABSTRACT. The filtrations on the James construction on spheres, $J_k\left(S^{2n}\right)$, have played a major role in the study of the double suspension $S^{2n-1} \to \Omega^2 S^{2n+1}$ and have been used to get information about the homotopy groups of spheres and Moore spaces and to construct product decompositions of related spaces. In this paper we calculate $H_*\left(\Omega J_k\left(S^{2n}\right);\mathbb{Z}/p\mathbb{Z}\right)$ for odd primes p. When k has the form p^t-1, the result is well known, but these are exceptional cases in which the homology has polynomial growth. We find that in general the homology has exponential growth and in some cases also has higher p-torsion. The calculations are applied to construct a p-local product decomposition of $\Omega J_k\left(S^{2n}\right)$ for $k < p^2-p$ which demonstrates a mod p homotopy exponent in these cases.

0. INTRODUCTION

One of the major problems in homotopy theory is to determine the properties of spaces of the form ΩX, where X is a simply connected finite complex. A dichotomy in the rational homotopy of such spaces has been observed by Felix, Halperin, and Thomas [FHT], who showed that for such X, either $\pi_n(X) \otimes \mathbb{Q}$ is 0 for all sufficiently large n or else its dimensions grow exponentially with n. Expressed in terms of homology rather than homotopy groups, this translates to saying that either $\dim H_n(\Omega X;\mathbb{Q})$ has polynomial growth or it has exponential growth. They introduced the terminology "elliptic" for the first type of space and "hyperbolic" for the second. Independently, John Moore observed that this distinction between elliptic and hyperbolic appears to be reflected in the properties of the torsion homotopy groups of the space as well. He conjectured that an elliptic complex should have a homotopy exponent for each prime while a hyperbolic complex should not have a homotopy exponent for any prime (cf. [S3]). That is, given a prime p, there should exist an r such that $p^r\big(p\text{-torsion } \pi_n(X)\big) = 0$ for all n if and only if X is elliptic. Given a prime p, spaces can be further differentiated according to whether or not the mod-p homology of their loop space has polynomial growth or not. We say that X is *mod-p elliptic* if $H_n\left(\Omega X;\mathbb{Z}/p\mathbb{Z}\right)$ has polynomial growth and *mod-p hyperbolic* otherwise. It is not known whether the mod-p homology of mod-p hyperbolic spaces must have exponential growth. Of course if X is hyperbolic, then it is mod p hyberbolic for each prime p, but an elliptic space may be mod p hyperbolic as illustrated by the mod p Moore space $S^{k-1} \cup_p e^k$ which we will write as $P^k(p)$.

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The Moore conjecture has been verified for only a handful of spaces, and with the exception of the Moore space $P^k(p)$ all of the elliptic spaces for which it has been verified for a given prime p are also mod-p elliptic at that prime. One of the purposes of this paper is to demonstrate a mod p homotopy exponent for certain elliptic spaces, many of which are mod-p hyperbolic.

Let $J_k(X)$ denote the k-th filtration of the James construction, $J(X)$, on X, where X is a pointed topological space with basepoint $*$. Explicitly, $J_k(X) = X^k/\sim$, where

$$(x_1, \dots, x_{j-1}, *, x_j, x_{j+1}, \dots, x_{k-1}) \sim (x_1, \dots, x_{j-1}, x_j, *, x_{j+1}, \dots, x_{k-1}).$$

Theorem (James). *If X is a connected CW complex, then*

1) $J(X) \approx \Omega\Sigma X$;

2) $\Sigma J_k(X) \approx \bigvee_{j=0}^{k} \Sigma X^{(j)}$.

The spaces $J_k\left(S^{2n}\right)$ have played a major role in the study of the double suspension map $E^2 : S^{2n-1} \to \Omega^2 S^{2n+1}$ and in determination of properties of homotopy groups of spheres and Moore spaces including construction of product decompositions of related spaces. After localization at a prime p the most tractible of these spaces are those for k of the form $p^t - 1$. Many properties of these spaces are known, including the fact that $\Omega J_{p^t-1}\left(S^{2n}\right)$ has a mod-p homotopy exponent for $t \geq 0$. However these are exceptional cases which are mod p elliptic.

In the first section of this paper we will calculate the mod p homology of $\Omega J_k\left(S^{2n}\right)$ for odd primes p. One of the interesting features of this homology is the appearance of higher p-torsion in some cases. As we shall see, although $J_k\left(S^{2n}\right)$ is elliptic for all k, for each prime p it is mod-p hyperbolic for "most" k. In section 2 we will obtain a homotopy decomposition of $\Omega J_k\left(S^{2n}\right)$ after localization at p in the cases $k < p^2 - p$ and use it to show that $J_k\left(S^{2n}\right)$ has a mod-p homotopy exponent for such k. While analogous decompositions may exist for $k \geq p^2 - p$ (and some cases will be shown in a followup paper [S4]), the existence of non-trivial Steenrod operations (other than the Bockstein) precludes a decomposition of exactly the same form.

Throughout the rest of this paper, let p be an odd prime. We will be working exclusively with p-local spaces and will abuse notation by using X to mean $X_{(p)}$.

Our main decomposition theorem states

Theorem. *For k such that $p \leq k < p^2 - p$ and $k \not\equiv -1(p)$*

$$\begin{aligned} &\Omega J_k\left(S^{2n}\right) \approx F_2(n) \\ &\times \Omega\left(S^{2n(k+1)-1} \vee \bigvee_{j=0}^{\infty} P^{2n(q+1)p+j(2np-2)-1}(p) \vee \bigvee_{j=0}^{\infty} P^{2n(k+p+1)+j(2np-2)-2}(p)\right) \end{aligned}$$

where $F_2(n)$ is a mod-p elliptic space introduced in [S2] *which is known to have a homotopy exponent.*

The cases where $p \leq k < p^2 - p$ and $k \equiv -1(p)$ are elliptic, and in these cases we get the simpler result

Theorem. *For k such that $p \le k < p^2 - p$ and $k \equiv -1(p)$*

$$\Omega J_k\left(S^{2n}\right) \approx F_2(n) \times \Omega S^{2n(k+1)-1}.$$

1. Homology of $\Omega J_k\left(S^{2n}\right)$

In this section we calculate the mod-p homology of $\Omega J_k\left(S^{2n}\right)$ as a Hopf algebra by means of $H_*\left(\Omega J_k\left(S^{2n}\right);\mathbb{Z}/p\mathbb{Z}\right) = H\big(A(k,n);\mathbb{Z}/p\mathbb{Z}\big)$, where $A(k,n)$ is an Adams-Hilton model for $J_k\left(S^{2n}\right)$.

For a graded set S, let $\mathbb{T}\langle S\rangle$, $\mathbb{S}\langle S\rangle$, $\mathbb{L}\langle S\rangle$ and $\mathbb{L}_{ab}\langle S\rangle$ denote respectively the tensor algebra, free abelian graded algebra, and the free and free abelian Lie algebras on S with $\mathbb{Z}_{(p)}$ coefficients, and let $\mathbb{T}^{\mathbb{Z}/p\mathbb{Z}}\langle S\rangle$, $\mathbb{S}^{\mathbb{Z}/p\mathbb{Z}}\langle S\rangle$, $\mathbb{L}^{\mathbb{Z}/p\mathbb{Z}}\langle S\rangle$ and $\mathbb{L}_{ab}^{\mathbb{Z}/p\mathbb{Z}}\langle S\rangle$ denote their mod-p reductions. Let $\mathcal{U}$ denote the universal enveloping algebra functor. Thus $\mathbb{T}\langle S\rangle = \mathcal{U}\mathbb{L}\langle S\rangle$ and $\mathbb{S}\langle S\rangle = \mathcal{U}\mathbb{L}_{ab}\langle S\rangle$.

The following lemmas are standard.

Lemma 1.1. *Let $A = \mathbb{L}\langle a_1, a_2, \ldots\rangle$. Let $a_i' = u_i a_i + c_i$, where u_i is a unit in $\mathbb{Z}_{(p)}$, $|c_i| = |a_i|$, and $c_i \in [A,A]$. Then the induced Lie algebra homomorphism $\mathbb{L}\langle a_1', a_2', \ldots\rangle \to A$ is an isomorphism. Similarly if $B = \mathbb{L}\langle a_1, b_1, a_2, b_2, \ldots\rangle$ and $a_i' = u_i a_i + \lambda_i b_i + c_i$, where u_i is a unit in $\mathbb{Z}_{(p)}$, $\lambda_i \in \mathbb{Z}_{(p)}$, $|c_i| = |a_i|$, and $c_i \in [A,A]$, then the induced Lie algebra homomorphism $\mathbb{L}\langle a_1', b_1, a_2', b_2, \ldots\rangle \to B$ is an isomorphism.* □

Lemma 1.2. *Let $\mathbb{L}\langle S\rangle$ be a differential Lie algebra (DGL) and let $A = \mathbb{L}\langle S \cup \{a,b\}\rangle$ where $da = b$. Then the canonical inclusion $\mathbb{L}\langle S\rangle \to A$ and surjection $A \to \mathbb{L}\langle S\rangle$ induce inverse isomorphisms on homology.* □

It is well known (cf. [A]) that $A(k,n) = \mathcal{U}L_0(k,n)$ where

$$L_0(k,n) = \mathbb{L}\langle y_1, y_2, \ldots, y_k\rangle,$$

with $|y_m| = 2nm - 1$ and

$$d(y_m) = -\frac{1}{2}\sum_{i=1}^{m-1}\binom{m}{i}[y_i, y_{m-i}].$$

In the notation for this and other Lie algebras which will be introduced, we will omit k and n when there is no possibility of confusion. Conversely we might write $y_{m,n}$ for additional clarity when more than one n is under consideration. Write $k = c_0 + c_1 p + c_2 p^2 + \ldots + c_t p^t$. Set $k_0 = k$, $k_1 = (k_0 - c_0)/p$, $k_2 = (k_1 - c_1)/p$, …, $k_t = c_t$.

Let L_1 denote the DGL kernel of the surjection $L_0 \to \mathbb{L}_{ab}\langle y_1\rangle$. Then

$$L_1 = \mathbb{L}\langle z_2, y_2, z_3, y_3, \ldots, z_k, y_k, z_{k+1}\rangle,$$

where $z_m = [y_1, y_{m-1}]$. For degree reasons the algebraic Serre spectral sequence for the short exact sequence of differential Hopf algebras $\mathcal{U}L_1 \to \mathcal{U}L_0 \to \mathcal{U}\mathbb{L}_{ab}\langle y_1\rangle$ collapses to give

$$\begin{aligned} H_*\left(\mathcal{U}L_0\right) &\cong H_*\left(\mathcal{U}L_1\right) \otimes H_*\left(\mathcal{U}\mathbb{L}_{ab}\langle y_1\rangle\right) \\ &\cong H_*\left(\mathcal{U}L_1\right) \otimes \mathbb{S}\langle y_1\rangle \end{aligned}$$

as coalgebras.

We wish to change basis for L_1, using Lemma 1.1, so as to be able to pair off some generators under d and be able to apply Lemma 1.2.

Let $\nu_p(m)$ or simply $\nu(m)$ denote the largest integer such that $p^{\nu_p(m)}$ divides m. For $m \le k$ let z'_m be $1/p^{\nu(m)}$ times the sum of those terms in the defining expression for dy_m indexed by integers j such that j and $m-j$ are not both divisible by p.

Explicitly,

$$z'_m = -\frac{1}{2} \sum_{\substack{j=1, \\ j\not\equiv 0(p) \ or \ m-j\not\equiv 0(p)}}^{m-1} \frac{1}{p^{\nu(m)}} \binom{m}{j} [y_j, y_{m-j}].$$

Thus

$$\begin{aligned} z'_m &= -\frac{m}{p^{\nu(m)}} z_m - \frac{1}{2} \sum_{\substack{j=2, \\ j\not\equiv 0(p) \ or \ m-j\not\equiv 0(p)}}^{m-2} \frac{1}{p^{\nu(m)}} \binom{m}{j} [y_j, y_{m-j}] \\ &= -\frac{m}{p^{\nu(m)}} z_m + [L_1, L_1], \end{aligned} \tag{1}$$

where here (and elsewhere) $x + [A, B]$ is used to indicate an element congruent to x modulo the subspace generated by $\{[a, b] \mid a \in A, b \in B\}$.

Although there is no y_{k+1} in $L_1(k, n)$, the above expression for z'_m does make sense when $m = k+1$, so we use it to extend the definition of z'_m to the case $m = k+1$. Since $m/p^{\nu(m)}$ is a unit in $\mathbb{Z}_{(p)}$, we can use Lemma 1.1 to change basis and write

$$L_1 = \mathbb{L}\langle z'_2, y_2, z'_3, y_3, \ldots, z'_k, y_k, z'_{k+1}\rangle.$$

If $m \not\equiv 0(p)$, then all terms in the expression for dy_m are terms in z'_m and so $dy_m = z'_m$ in this case. Notice also that $dy_p = pz'_p$. Let

$$L'_1 = \mathbb{L}\langle z'_p, y_p, z'_{2p}, y_{2p}, \ldots, z'_{k_1 p}, y_{k_1 p}, z'_{k+1}\rangle.$$

The above definitions imply that L'_1 is closed under the differential and Lemma 1.2 shows that the inclusion of DGL's $L'_1 \hookrightarrow L_1$ induces an isomorphism on homology. Let

$$K_1 = \mathbb{L}\langle z'_p, y_p, z'_{2p}, y_{2p}, \ldots, z'_{k_1 p}, y_{k_1 p}\rangle.$$

K_1 is a sub-Lie algebra of L'_1.

If $k < p$, then $L'_1 = \mathbb{L}\langle z'_{k+1}\rangle$ and so $H_*\left(\Omega J_k\left(S^{2n}\right)\right)$ is given by $H_*\left(\Omega J_k\left(S^{2n}\right)\right) = \mathbb{S}\langle y_1, z'_{k+1}\rangle$. We now suppose that $k \ge p$. Let L_2 denote the DGL kernel of the surjection $L'_1 \to \mathbb{L}_{ab}\langle z'_p, y_p\rangle$. L_2 becomes a $\mathcal{U}\mathbb{L}_{ab}\langle z'_p, y_p\rangle$-module under the action induced by $v \cdot w = [v, w]$ for $v \in \mathbb{L}_{ab}\langle z'_p, y_p\rangle$, $w \in L_2$. Set $a_m = [y_p, y_{m-p}]$ and $b_m = [y_p, z'_{m-p}]$. Then $L_2 = \mathbb{L}\langle B_2\rangle$, where B_2 is a basis for the free $\mathcal{U}\mathbb{L}_{ab}\langle z'_p\rangle$-module on

$$\begin{aligned} \{&b_{2p}, a_{2p}, z'_{2p}, y_{2p}, b_{3p}, a_{3p}, z'_{3p}, y_{3p}, \ldots b_{k_1 p}, a_{k_1 p}, \\ &z'_{k_1 p}, y_{k_1 p}, z'_{k+1}, b_{(k_1+1)p}, a_{(k_1+1)p}, b_{k+p+1}\}. \end{aligned}$$

After reduction modulo p the algebraic Serre spectral sequence again collapses to give the coalgebra decomposition

$$H_*\left(\mathcal{U}L_1;\mathbb{Z}/p\mathbb{Z}\right) \cong H_*\left(\mathcal{U}L_2;\mathbb{Z}/p\mathbb{Z}\right) \otimes \mathbb{S}^{\mathbb{Z}/p\mathbb{Z}}\langle z'_p, y_p\rangle$$

so that

$$H_*\left(\Omega J_k\left(S^{2n}\right);\mathbb{Z}/p\mathbb{Z}\right) \cong H_*\left(\mathcal{U}L_2;\mathbb{Z}/p\mathbb{Z}\right) \otimes \mathbb{S}^{\mathbb{Z}/p\mathbb{Z}}\langle y_1, z'_p, y_p\rangle$$

as coalgebras.

Our next goal is to rechoose our generators of L_2 in such a way as to pair off more generators under d and again apply Lemma 1.2 to produce a sub-Lie algebra L'_2 with $L'_2 \hookrightarrow L_2$ inducing an isomorphism on homology. We will then show how to relate the homology of L'_2 to that of $L'_1(k_1, np)$, thus obtaining the homology of L'_2 in a recursive sense.

For q such that $q \leq k$ and m divisible by p let a'_q be $1/p^{\nu(q)-1}$ times the sum of those terms indexed in the defining expression for dy_q by integers j such that j and $q-j$ are not both divisible by p^2. Explicitly,

$$\begin{aligned} a'_{mp} = &-\frac{1}{2}\sum_{\substack{j=1,\\ j\not\equiv 0(p)}}^{mp-1} \frac{1}{p^{\nu(mp)-1}}\binom{mp}{j}[y_j, y_{mp-j}] \\ &-\frac{1}{2}\sum_{\substack{j=1,\\ j\not\equiv 0(p)\ or\ m-j\not\equiv 0(p)}}^{m-1} \frac{1}{p^{\nu(mp)-1}}\binom{mp}{jp}[y_{jp}, y_{mp-jp}]. \end{aligned}$$

Thus

$$\begin{aligned} a'_{mp} &= pz'_{mp} - \frac{1}{p^{\nu(mp)-1}}\binom{mp}{p}a_{mp} \\ &\quad -\frac{1}{2}\sum_{\substack{j=2,\\ j\not\equiv 0(p)\ or\ m-j\not\equiv 0(p)}}^{m-2} \frac{1}{p^{\nu(m)-1}}\binom{mp}{jp}[y_{jp}, y_{m-jp}] \\ &= pz'_{mp} - \frac{1}{p^{\nu(mp)-1}}\binom{mp}{p}a_{mp} + [L_2, L_2]. \end{aligned} \tag{2}$$

Note that $\left(1/p^{\nu(mp)-1}\right)\binom{mp}{p}$ is a unit modulo p and that if $m \not\equiv 0(p)$, then the definition implies that $dy_{mp} = a'_{mp}$.

Differentiating

$$dy_{mp} = p^{\nu(mp)}z'_{mp} - \frac{1}{2}\sum_{i=1}^{m-1}\binom{mp}{ip}[y_{ip}, y_{mp-ip}]$$

yields

$$\begin{aligned}
dz'_{mp} &= \frac{1}{2p^{\nu(mp)}}\sum_{i=1}^{m-1}\binom{mp}{ip}d[y_{ip}, y_{mp-ip}] \\
&= \frac{1}{p^{\nu(mp)}}\binom{mp}{p}d[y_p, y_{mp-p}] + \frac{1}{2p^{\nu(mp)}}\sum_{i=2}^{m-2}\binom{mp}{ip}d[y_{ip}, y_{mp-ip}] \\
&= -\frac{1}{p^{\nu(mp)}}\binom{mp}{p}\big([pz'_p, y_{mp-p}] - [y_p, dy_{mp-p}]\big) + [L_2, L_2] \\
&= \frac{1}{p^{\nu(mp)}}\binom{mp}{p}[y_p, dy_{mp-p}] - \frac{p}{p^{\nu(mp)}}\binom{mp}{p}[z'_p, y_{mp-p}] + [L_2, L_2].
\end{aligned}$$

Using the fact that L_2 is a $\mathcal{U}\mathbb{L}_{ab}\langle z'_p, y_p\rangle$-module, or directly from the Jacobi identity we see that $\big[y_p, [L_2, L_2]\big] \subset [L_2, L_2]$, and so $\big[y_p, [y_{ip}, y_{jp}]\big]$ belongs to $[L_2, L_2]$ for all i and j. Thus

$$[y_p, dy_{mp-p}] = [y_p, p^{\nu(mp-p)}z'_{mp-p}] + [L_2, L_2] = p^{\nu(mp-p)}b_{mp} + [L_2, L_2].$$

Therefore

$$dz'_{mp} = \frac{p^{\nu(mp-p)}}{p^{\nu(mp)}}\binom{mp}{p}b_{mp} - \frac{p}{p^{\nu(mp)}}\binom{mp}{p}[z'_p, y_{mp-p}] + [L_2, L_2]. \tag{3}$$

Notice that if $m \not\equiv 1(p)$, then $\big(p^{\nu(mp-p)}/p^{\nu(mp)}\big)\binom{mp}{p} = \big(p/p^{\nu(mp)}\big)\binom{mp}{p}$ is a unit modulo p.

Turning now to the case $m \equiv 1(p)$, write $m = qp + 1$. From (3),

$$\begin{aligned}
dz'_{qp^2+p} &= \frac{p^{\nu(qp^2)}}{p^{\nu(qp^2+p)}}\binom{qp^2+p}{p}b_{qp^2+p} - \frac{p}{p^{\nu(qp^2+p)}}\binom{qp^2+p}{p}[z'_p, y_{qp^2}] + [L_2, L_2] \\
&= \frac{p^{\nu(qp^2)}}{p}\binom{qp^2+p}{p}b_{qp^2+p} - \binom{qp^2+p}{p}[z'_p, y_{qp^2}] + [L_2, L_2]. \qquad (4)
\end{aligned}$$

Note that $\binom{qp^2+p}{p}$ is a unit modulo p. Our intention is to use these formulas to justify replacing the generators $(z'_p)^j \cdot b_{mp}$ by $d\big((z'_p)^j \cdot dz'_{mp}\big)$ for $m \not\equiv 1(p)$ and $j \geq 0$ and replacing $(z'_p)^j \cdot y_{qp^2}$ by $d\Big((z'_p)^{j-1} \cdot z'_{qp^2+p}\Big)$, for $j \geq 1$.

Also,

$$\begin{aligned}
db_{qp^2+p} &= d[y_p, z'_{qp^2}] \\
&= [pz'_p, z'_{qp^2}] - [y_p, dz'_{qp^2}] \\
&= p[z'_p, z'_{qp^2}] - \left[y_p, \frac{p^{\nu(qp^2-p)}}{p^{\nu(qp^2)}}\binom{qp^2}{p}b_{qp^2} - \frac{p}{p^{\nu(qp^2)}}\binom{qp^2}{p}[z'_p, y_{qp^2-p}]\right] + [L_2, L_2]
\end{aligned}$$

using the earlier expression for dz'_{qp^2} and the fact that $\big[y_p, [L_2, L_2]\big] \subset [L_2, L_2]$. The Jacobi identity gives $[y_p, b_{qp^2}] = \Big[y_p, [y_p, z'_{qp^2-p}]\Big] = (1/2)[a_{2p}, z'_{qp^2-p}] \in [L_2, L_2]$. Similarly $\big[y_p, [z'_p, y_{qp^2-p}]\big] = [z'_p, a_{qp^2}] + [L_2, L_2]$. Therefore

$$db_{qp^2+p} = p[z'_p, z'_{qp^2}] - \frac{p}{p^{\nu(qp^2)}}\binom{qp^2}{p}[z'_p, a_{qp^2}] + [L_2, L_2]. \tag{5}$$

Noting that $\left(p/p^{\nu(qp^2)}\right)\binom{qp^2}{p}$ is a unit modulo p, we shall use this formula to replace $(z'_p)^j \cdot a_{qp^2}$ by $d\left((z'_p)^{j-1} \cdot b_{qp^2+p}\right)$ for $j \geq 1$.

Combining (1)–(5) gives

$$L_2 = \mathbb{L}\langle a'_{p^2}, y_{p^2}, a'_{2p^2}, y_{2p^2}, \dots, a'_{k_2p^2}, y_{k_2p^2} \cup C_2 \cup D_2\rangle$$

where C_2 is a basis for the free $\mathcal{U}\mathbb{L}_{ab}\langle z'_p\rangle$-module on

$$\left\{\{dy_{mp}, y_{mp}\}_{2\leq m\leq k_1, m\not\equiv 0(p)} \cup \{dz'_{mp}, z'_{mp}\}_{2\leq m\leq k_1} \cup \{db_{qp^2+p}, b_{qp^2+p}\}_{1\leq q\leq k_2}\right\}$$

and D_2 is a basis for the free $\mathcal{U}\mathbb{L}_{ab}\langle z'_p\rangle$-module on

$$\left\{z'_{k+1}, c_{(k_1+1)p}, a_{(k_1+1)p}, b_{k+p+1}\right\}$$

with

$$c_{(k_1+1)p} = \begin{cases} b_{(k_1+1)p}, & \text{if } k_1 \not\equiv 0(p); \\ z'_p \cdot y_{k_1p}, & \text{if } k_1 \equiv 0(p). \end{cases}$$

Let

$$K_2 = \mathbb{L}\langle a'_{p^2}, y_{p^2}, a'_{2p^2}, y_{2p^2}, \dots a'_{k_2p^2}, y_{k_2p^2}\rangle,$$

and let I_2 be the Lie ideal of L_2 generated by C_2.

Lemma 1.3.
1)$[y_p, K_2] \subset I_2$.
2)$[z'_p, K_2] \subset I_2$.

Proof. 1) Examination of (2) shows that the terms appearing in that formula as $[L_2, L_2]$ always lie in I_2 and in fact lie in $[I_2, I_2]$ when $m \equiv 0(p)$. The Jacobi identity implies that $\left[y_p, [I_2, I_2]\right] \subset I_2$ and so

$$\begin{aligned} &\equiv \left[y_p, pz'_{qp^2} - \frac{1}{p^{\nu(qp^2)-1}}\binom{qp^2}{p}a_{qp^2}\right] \\ &= p[b_{qp^2+p}] - \frac{1}{p^{\nu(qp^2)-1}}\binom{qp^2}{p}\left[y_p, [y_p, y_{qp^2-p}]\right] \\ &\equiv \frac{1}{2p^{\nu(qp^2)-1}}\binom{qp^2}{p}[y_{qp^2-p}, a_{2p}] \\ &\equiv 0 \qquad \bmod I_2. \end{aligned}$$

Also

$$\begin{aligned} \binom{qp^2+p}{p}[y_p, y_{qp^2}] &= \binom{qp^2+p}{p}a_{qp^2+p} \\ &\equiv -a'_{qp^2+p} + pz'_{qp^2+p} \\ &\equiv -dy_{qp^2+p} + pz'_{qp^2+p} \\ &\equiv 0 \qquad \bmod I_2 \end{aligned}$$

and so $[y_p, y_{qp^2}] \in I_2$.

2) Let x belong to K_2. Since I_2 is a differential ideal ($dI_2 \subset I_2$) and $[y_p, x] \in I_2$ by (1), differentiating gives $p[z'_p, x] - [y_p, dx] \in I_2$. Since $[y_p, dx] \in [y_p, K_2] \subset I_2$, this gives $[z'_p, x] \in I_2$. $\square$

Since I_2 is acylic, the preceding lemma implies that the inclusions $[y_p, K_2] \hookrightarrow L_2$ and $[z_p, K_2] \hookrightarrow L_2$ induce the zero map on homology. The key step in the recursion is showing that $K_2(k,n) \cong K_1(k_1, np)$.

Write $y^{\#}_m$ and ${z'}^{\#}_m$ for the generators $y_{m,np}$ and $z'_{m,np}$ of $K_1(k_1, np)$. Let

$$u(m) = \frac{m!(p!)^m}{(mp)!}.$$

The following lemma is easily checked.

Lemma 1.4.

a) *$u(m)$ is a unit modulo p for all m.*
b) *$\binom{mp}{jp}u(m) = \binom{m}{j}u(j)u(m-j)$ for all j and m.* □

Insight into the definition of $u(m)$ is as follows. Let $H : \Omega J\left(S^{2n}\right) \to \Omega J\left(S^{2np}\right)$ denote the pth Hopf-invariant map. Toda [T] checked that for all m, H induces an isomorphism on $H_{2nm}(\ ;\mathbb{Z}_{(p)})$ by showing that it induces multiplication by $1/u(m)$ in degree $2nm$.

Define $g : K_1(k_1, np) \to K_2(k,n)$ by $g(y^{\#}_{mp}) = u(mp)y_{mp^2}$ and $g({z'}^{\#}_{mp}) = u(mp)a'_{mp^2}$.

Theorem 1.5. *g is an isomorphism of differential graded Lie algebras.*

Proof. It is clear that g is an isomorphism of graded Lie algebras, but we must check that g commutes with the differentials.

$$d^{\#}(y^{\#}_{mp}) = p^{\nu(mp)}{z'}^{\#}_{mp} - \frac{1}{2}\sum_{j=1}^{m-1}\binom{mp}{jp}[y^{\#}_{jp}, y^{\#}_{mp-jp}] \tag{6}$$

and

$$\begin{aligned} d(y_{mp^2}) &= p^{\nu(mp^2)-1}a'_{mp^2} - \frac{1}{2}\sum_{j=1}^{m-1}\binom{mp^2}{jp^2}[y_{jp^2}, y_{mp^2-jp^2}] \\ &= p^{\nu(mp)}a'_{mp^2} - \frac{1}{2}\sum_{j=1}^{m-1}\binom{mp^2}{jp^2}[y_{jp^2}, y_{mp^2-jp^2}]. \end{aligned} \tag{7}$$

Therefore

$$\begin{aligned} gd^{\#}(y^{\#}_{mp}) &= p^{\nu(mp)}g({z'}^{\#}_{mp}) - \frac{1}{2}\sum_{j=1}^{m-1}\binom{mp}{jp}[g(y^{\#}_{jp}), g(y^{\#}_{mp-jp})] \\ &= u(mp)p^{\nu(mp)}a'_{mp^2} - \frac{1}{2}\sum_{j=1}^{m-1}u(jp)u\big((m-j)p\big)\binom{mp}{jp}[y_{jp^2}, y_{mp^2-jp^2}] \\ &= u(mp)p^{\nu(mp)}a'_{mp^2} - u(mp)\frac{1}{2}\sum_{j=1}^{m-1}\binom{mp^2}{jp^2}[y_{jp^2}, y_{mp^2-jp^2}] \\ &= u(mp)dy_{mp^2} \\ &= dg(y^{\#}_{mp}). \end{aligned}$$

Differentiating (6) and solving for $d^{\#}({z'}^{\#}_{mp})$ gives

$$p^{\nu(mp)}d^{\#}({z'}^{\#}_{mp}) = \frac{1}{2}\sum_{j=1}^{m-1}\binom{mp}{jp}d^{\#}([y^{\#}_{jp}, y^{\#}_{mp-jp}])$$

and (7) gives a similar expression for $d(a'_{mp^2})$. Therefore

$$\begin{aligned}
p^{\nu(mp)}gd^{\#}({z'}^{\#}_{mp}) =&\frac{1}{2}\sum_{j=1}^{m-1}\binom{mp}{jp}gd^{\#}([y^{\#}_{jp}, y^{\#}_{mp-jp}])\\
=&\frac{1}{2}\sum_{j=1}^{m-1}\binom{mp}{jp}d^{\#}g([y^{\#}_{jp}, y^{\#}_{mp-jp}])\\
=&\frac{1}{2}\sum_{j=1}^{m-1}\binom{mp}{jp}u(jp)u\big((m-j)p\big)d([y_{jp^2}, y_{mp^2-jp^2}])\\
=&\frac{1}{2}\sum_{j=1}^{m-1}\binom{mp^2}{jp^2}u(mp)d([y_{jp^2}, y_{mp^2-jp^2}])\\
=&u(mp)p^{\nu(mp)}a_{mp^2}\\
=&p^{\nu(mp)}dg({z'}^{\#}_{mp})
\end{aligned}$$

where we have made use of the previously verified result that $gd^{\#} = dg$ on $y^{\#}_{mp}$. Therefore $gd^{\#} = dg$ on all of the Lie algebra generators. □

Remark. Although the situation is not exactly identical, this calculation is essentially the same as that in [A, proof of Lemma 2.6b].

We continue choosing our new generators for L_2. Let $h = g^{-1} : K_2(k,n) \to K_1(k_1,np)$. We next choose $c'_{(k_1+1)p}$ and $a'_{(k_1+1)p}$ to replace $c_{(k_1+1)p}$ and $a_{(k_1+1)p}$ and extend the domain of $h : K_2(k,n) \to K_1(k_1,np) \hookrightarrow L_1(k_1,np)$ to include z'_{k+1} and these elements.

The definition of a'_{mp} we used when $mp \le k$ will not make sense when $m = k_1+1$ since there does not exist $y_{(k_1+1)p}$ in $L_2(k,n)$. However there is such an element in $L_2(\infty,n)$, and we will denote it as $\hat{y}_{(k_1+1)p}$. In $L_2(\infty,n)$ write

$$d\hat{y}_{(k_1+1)p} = p^t\hat{z}_{(k_1+1)p} + p^{t-1}\alpha + \beta,$$

where $t = \nu\big((k_1+1)p\big)$,

$$\hat{z}_{(k_1+1)p} = -\frac{1}{2}\sum_{\substack{j=1,\\ j\not\equiv 0(p)}}^{m-1}\frac{1}{p^t}\binom{(k_1+1)p}{j}[y_j, y_{(k_1+1)p-j}],$$

$$\alpha = -\frac{1}{2}\sum_{\substack{j=1,\\ j\not\equiv 0(p)\ or\ k_1+1-j\not\equiv 0(p)}}^{k_1}\frac{1}{p^{t-1}}\binom{(k_1+1)p}{jp}[y_{jp}, y_{(k_1+1)p-jp}],$$

and β is the sum of the remaining terms. If $k_1 \not\equiv -1(p)$, then $t = 1$ and $\beta = 0$. If $k_1 \equiv -1(p)$, then $k_1 + 1 = (k_2 + 1)p$ and

$$\beta = -\frac{1}{2}\sum_{j=1}^{k_2}\binom{(k_1+1)p}{jp^2}[y_{jp^2}, y_{(k_1+1)p-jp^2}].$$

Notice that α and β lie in $L_2(k, n)$. Since differentiating gives

$$0 = p^t d\hat{z}_{(k_1+1)p} + p^{t-1}d\alpha + d\beta,$$

we see that $d\beta$ is divisible by p^{t-1} and $d\alpha + d\beta/p^{t-1}$ is divisible by p.

Lemma 1.6. $d\beta = \dfrac{p^{t-1}}{u\big((k_1+1)p\big)} g d{z'}^{\#}_{k_1+1}$.

Proof. If $k_1 \not\equiv -1(p)$, then both sides are zero. For $k_1 \equiv -1(p)$,

$$dy^{\#}_{k_1+1} = p^{t-1}{z'}^{\#}_{k_1+1} + \frac{1}{2}\sum_{j=1}^{k_2}\binom{k_1+1}{jp}[y^{\#}_{jp}, y^{\#}_{k_1+1-jp}]$$

and so

$$\begin{aligned} p^{t-1}gd{z'}^{\#}_{k_1+1} &= -\frac{1}{2}\sum_{j=1}^{k_2}\binom{k_1+1}{jp}u(jp)u\big((k_1+1)p - jp\big)d[y_{jp^2}, y_{(k_1+1)p-jp^2}] \\ &= -\frac{1}{2}\sum_{j=1}^{k_2}\binom{(k_1+1)p}{jp^2}u\big((k_1+1)p\big)d[y_{jp^2}, y_{(k_1+1)p-jp^2}] \\ &= u\big((k_1+1)p\big)d\beta \qquad \square \end{aligned}$$

Set $a'_{(k_1+1)p} = -\alpha$ and $c'_{(k_1+1)p} = d\hat{z}_{(k_1+1)p} = -(d\alpha + d\beta/p^{t-1})/p$. Thus

$$da'_{(k_1+1)p} = pc'_{(k_1+1)p} + \frac{gd{z'}^{\#}_{k_1+1}}{u\big((k_1+1)p\big)}. \tag{8}$$

From the definition,

$$a'_{(k_1+1)p} = \alpha = -\frac{1}{p^{t-1}}\binom{(k_1+1)p}{p}a_{(k_1+1)p} + [L_2, L_2],$$

and note that $\big(1/p^{t-1}\big)\binom{(k_1+1)p}{p}$ is a unit modulo p. Therefore we may replace $a_{(k_1+1)p}$ by $a'_{(k_1+1)p}$ in our basis for L_2. Differentiating gives

$$pc'_{(k_1+1)p} + \frac{gdz^{\#}_{k_1+1}}{u\big((k_1+1)p\big)} = -\frac{1}{p^{t-1}}\binom{(k_1+1)p}{p}da_{(k_1+1)p} + [L_2, L_2].$$

Since $gd{z'}^{\#}_{k_1+1} \in [L_2, L_2]$,

$$\begin{aligned} c'_{(k_1+1)p} &= -\frac{1}{p^t}\binom{(k_1+1)p}{p}([pz'_p, y_{k_1p}] - [y_p, dy_{k_1p}]) + [L_2, L_2] \\ &= -\frac{1}{p^t}\binom{(k_1+1)p}{p}([pz'_p, y_{k_1p}] - [y_p, p^{\nu(k_1p)}z'_{k_1p}]) + [L_2, L_2] \\ &= -\frac{1}{p^{t-1}}\binom{(k_1+1)p}{p}([z'_p, y_{k_1p}] - p^{\nu(k_1)}b_{(k_1+1)p}) + [L_2, L_2]. \end{aligned}$$

Note that $[z'_p, y_{k_1p}]$ appears in our basis when $k_1 \not\equiv 0(p)$, so this formula together with the fact that $\left(1/p^{t-1}\right)\binom{(k_1+1)p}{p}$ is a unit modulo p justifies the replacement of $c_{(k_1+1)p}$ by $c'_{(k_1+1)p}$ in our basis for L_2 both in the case where $k_1 \equiv 0(p)$ and the case $k_1 \not\equiv 0(p)$.

Let

$$M_2 = \mathbb{L}\langle a'_{p^2}, y_{p^2}, a'_{2p^2}, y_{2p^2}, \dots, a'_{k_2p^2}, y_{k_2p^2}, z'_{k+1}, c'_{(k_1+1)p}, a'_{(k_1+1)p}\rangle.$$

If $k \not\equiv -1(p)$, then $dz'_{k+1} = 0$. In the case $k \equiv -1(p)$, we have $k+1 = (k_1+1)p$, so this is the one case where $\hat{z}_{(k_1+1)p}$ lies in $L_2(k,n)$. In this case $\hat{z}_{(k_1+1)p} = z'_{(k_1+1)p}$ and so $dz'_{(k_1+1)p} = c'_{(k_1+1)p}$. Extend the definition of h to $\bar{h} : M_2(k,n) \to L'_1(k_1, np)$ by setting $\bar{h}(z'_{k+1}) = 0$, $\bar{h}(c'_{(k_1+1)p}) = 0$, and $\bar{h}(a'_{(k_1+1)p}) = \big(1/u\big((k_1+1)p\big)\big)\, {z'}^{\#}_{k_1+1}$. By the above calculations, $\bar{h}$ commutes with the differentials. Furthermore we can form a Lie algebra extension $\bar{g}$ of g to $L'_1(k_1, np)$ by setting

$$\bar{g}({z'}^{\#}_{k_1+1}) = \begin{cases} u\big((k_1+1)p\big)a'_{(k_1+1)p}, & \text{if } k \not\equiv -1(p); \\ u\big((k_1+1)p\big)(a'_{(k_1+1)p} - pz'_{k+1}), & \text{if } k \equiv -1(p) \end{cases} \tag{9}$$

to form a right inverse to $\bar{h}$. Formula (8) implies that this extension commutes with the differentials after reduction modulo p. The extra term we have included when $k \equiv -1(p)$ is not essential to the computation, but it makes the determination of the operation of the Bockstein easier. This splitting together with the fact that

$$dz'_{k+1} = \begin{cases} 0, & \text{if } k \not\equiv -1(p); \\ c'_{(k_1+1)p}, & \text{if } k \equiv -1(p) \end{cases}$$

shows that the mod-p homology of $\mathcal{U}M_2$ is given by

$$H_*\left(\mathcal{U}M_2(k,n); \mathbb{Z}/p\mathbb{Z}\right) = \begin{cases} \bar{g}H_*\left(\mathcal{U}L'_1(k_1,np); \mathbb{Z}/p\mathbb{Z}\right) \\ \qquad \coprod \mathbb{T}^{\mathbb{Z}/p\mathbb{Z}}\langle c'_{(k_1+1)p}, z'_{k+1}\rangle, & \text{if } k \not\equiv -1(p); \\ \bar{g}H_*\left(\mathcal{U}L'_1(k_1,np); \mathbb{Z}/p\mathbb{Z}\right), & \text{if } k \equiv -1(p). \end{cases} \tag{10}$$

At this point we have

$$L_2 = \mathbb{L}\langle a'_{p^2}, y_{p^2}, a'_{2p^2}, y_{2p^2}, \dots, a'_{k_2p^2}, y_{k_2p^2}, z'_{k+1}, c'_{(k_1+1)p}, a'_{(k_1+1)p} \cup C_2 \cup D'_2\rangle$$

where C_2 is as before and D'_2 is a basis for the free $\mathcal{U}\mathbb{L}_{ab}\langle z'_p\rangle$-module on

$$\left\{ z'_p \cdot z'_{k+1}, z'_p \cdot c'_{(k_1+1)p}, z'_p \cdot a'_{(k_1+1)p}, b_{k+p+1} \right\}.$$

As graded Lie algebras, we can rewrite this as

$$L_2 = M_2 \coprod \mathbb{L}\langle C_2\rangle \coprod \mathbb{L}\langle D_2'\rangle;$$

however, this need not be true as differential graded Lie algebras as $\mathbb{L}\langle D_2'\rangle$ might not be closed under the differential. Our final move is to replace $z_p' \cdot a'_{(k_1+1)p}$ by an element $w_{(k_1+2)p}$ so as to rectify this defect.

We consider first the case $k \equiv -1(p)$. In this case $dz'_{k+1} = c'_{(k_1+1)p}$ and so $d(z_p' \cdot z'_{(k_1+1)p}) = z_p' \cdot c'_{(k_1+1)p}$. Set $w_{(k_1+2)p} = db_{k+p+1}$. Explicitly,

$$\begin{aligned}
w_{(k_1+2)p} &= db_{k+p+1}\\
&= d[y_p, z'_{k+1}]\\
&= p[z_p', z'_{k+1}] - [y_p, c'_{(k_1+1)p}]\\
&= p[z_p', z'_{k+1}] + [y_p, \frac{d\alpha}{p}] + [y_p, \frac{gdz^{\#}_{k_1+1}}{pu\big((k_1+1)p\big)}]\\
&= p[z_p', z'_{k+1}]\\
&\quad + \frac{1}{p}\left[y_p, d\left(-\frac{1}{2}\sum_{\substack{j=1,\\ j\not\equiv 0(p)\ \text{or}\ k_1+1-j\not\equiv 0(p)}}^{k_1} \frac{1}{p^{t-1}}\binom{(k_1+1)p}{jp}[y_{jp}, y_{(k_1+1)p-jp}]\right)\right]\\
&\quad + \big[y_p, [L_2, L_2]\big]\\
&= p[z_p', z'_{k+1}] - \frac{1}{2p^t}\left[y_p, d\left(\binom{(k_1+1)p}{p}2[y_p, y_{k_1p}]\right.\right.\\
&\qquad \left.\left. + \sum_{\substack{j=2,\\ j\not\equiv 0(p)\ \text{or}\ k_1+1-j\not\equiv 0(p)}}^{k_1-1} \binom{(k_1+1)p}{jp}[y_jp, y_{(k_1+1)p-jp}]\right)\right] + [L_2, L_2]\\
&= p[z_p', z'_{k+1}] - \frac{1}{p^t}\binom{(k_1+1)p}{p}[y_p, d[y_p, y_{k_1p}]] + [L_2, L_2]\\
&= p[z_p', z'_{k+1}] - \frac{1}{p^t}\binom{(k_1+1)p}{p}\big([y_p, [pz_p', y_{k_1p}]] - [y_p, [y_p, dy_{k_1p}]]\big) + [L_2, L_2]\\
&= p[z_p', z'_{k+1}] - \frac{1}{p^t}\binom{(k_1+1)p}{p}\Big(\big[pz_p', [y_p, y_{k_1p}]\big] + \big[y_{k_1p}, [y_p, pz_p']\big]\\
&\qquad\qquad + \frac{1}{2}[a_{2p}, dy_{k_1p}]\Big) + [L_2, L_2]\\
&= p[z_p', z'_{k+1}] - \frac{1}{p^t}\binom{(k_1+1)p}{p}\big([pz_p', a_{(k_1+1)p}] + [y_{k_1p}, pb_{2p}]\big) + [L_2, L_2]\\
&= p[z_p', z'_{k+1}] + [z_p', a'_{(k_1+1)p}] + [L_2, L_2].
\end{aligned}$$

Thus $w_{(k_1+2)p} = db_{k+p+1}$ is a valid replacement for $z_p' \cdot a'_{(k_1+1)p}$ in this case.

Finally when $k \not\equiv -1(p)$, we will choose $w_{(k_1+2)p}$ in such a way that $dw_{(k_1+2)p} = pz_p' \cdot c_{(k_1+1)p}$. Since $z_p' \cdot gd^{\#} z'^{\#}_{k_1+1}$ is a cycle in $[z_p', K_2]$, its image in L_2 is a boundary by Lemma 1.3. Choose γ such that $d\gamma = z_p' \cdot gd^{\#} z'^{\#}_{k_1+1}$. Since the bracket length of $z'_p \cdot gd^{\#}z'^{\#}_{k_1+1}$ is greater than 2, we see that $\gamma \in [L_2, L_2]$. Set $w_{(k_1+2)p} = z'_p \cdot a'_{(k_1+1)p} - \gamma/u\big((k_1+1)p\big)$. Then $w_{(k_1+2)p}$ is a valid replacement for $z'_p \cdot a'_{(k_1+1)p}$ and $dw_{(k_1+2)p} = pz'_p \cdot c'_{(k_1+1)p}$.

Lemma 1.7. *For $k \not\equiv -1(p)$, $w_{(k_1+2)p}$ is homologous to $[y_p, c'_{(k_1+1)p}]$.*

Proof. Write $a \sim b$ to mean a is homologous to b. By construction $w_{(k_1+2)p} \sim [z'_p, a'_{(k_1+1)p}]$. For all x, $[y_p, dx] \sim p[z'_p, x]$ since $d[y_p, x] = p[z'_p, x] - [y_p, dx]$. Thus for $x \in K_2$, $[y_p, dx] \sim [z'_p, x] \sim 0$ by Lemma 1.3. By definition $c'_{(k_1+1)p} = da'_{(k_1+1)p}/p$ $-d\beta/p^{\nu((k_1+1)p)}$ where $\beta \in K_2$. Therefore $[y_p, c'_{(k_1+1)p}] \sim [pz'_p, a'_{(k_1+1)p}/p] \sim [z'_p, a'_{(k_1+1)p}]$. □

After this replacement in both cases we get a decomposition as DGL's

$$L_2 = M_2 \coprod C_2 \coprod N_2$$

where M_2 and C_2 are as above and $N_2 = \mathbb{L}\langle D''_2\rangle$ where D''_2 is a basis for the free $\mathcal{U}\mathbb{L}_{ab}\langle z'_p\rangle$-module on

$$\left\{z'_p \cdot z'_{k+1}, z'_p \cdot c'_{(k_1+1)p}, w_{(k_1+2)p}, b_{k+p+1}\right\},$$

with the differential given by

$$\left\{\begin{matrix} dw_{(k_1+2)p} = pz'_p \cdot c'_{(k_1+1)p} \\ db_{k+p+1} = pz'_p \cdot z'_{k+1} \end{matrix}\right\}, \qquad if\ k \not\equiv -1(p)$$

and

$$\left\{\begin{matrix} dz'_p \cdot z'_{k+1} = z'_p \cdot c'_{(k_1+1)p} \\ db_{k+p+1} = w_{(k_1+2)p} \end{matrix}\right\}, \qquad if\ k \equiv -1(p).$$

Applying Lemma 1.2 gives a mod p homology isomorphism

$$H_*\left(\mathcal{U}(M_2 \textstyle\coprod N'_2); \mathbb{Z}/p\mathbb{Z}\right) \to H_*\left(\mathcal{U}(M_2 \textstyle\coprod N_2); \mathbb{Z}/p\mathbb{Z}\right)$$

with

$$N'_2 = \begin{cases} N_2, & \text{if } k \not\equiv -1(p); \\ 0, & \text{if } k \equiv -1(p). \end{cases}$$

Thus combining this with (10) gives

$$\begin{aligned} &H_*\left(\mathcal{U}L'_2(k,n); \mathbb{Z}/p\mathbb{Z}\right) \\ &= \begin{cases} \bar{g}H_*\left(\mathcal{U}L'_1(k_1, np); \mathbb{Z}/p\mathbb{Z}\right) \\ \qquad \coprod \mathbb{T}^{\mathbb{Z}/p\mathbb{Z}}\Big\langle \{z'^{\,j}_p \cdot z'_{k+1}\}_{j\geq 0}, \\ \{z'^{\,j}_p \cdot c'_{(k_1+1)p}\}_{j\geq 0}, \{z'^{\,j}_p \cdot w_{(k_1+2)p}\}_{j\geq 0}, \{z'^{\,j}_p \cdot b_{k+p+1}\}_{j\geq 0}\Big\rangle, & \text{if } k \not\equiv -1(p); \\ \bar{g}H_*\left(\mathcal{U}L'_1(k_1, np); \mathbb{Z}/p\mathbb{Z}\right), & \text{if } k \equiv -1(p) \end{cases} \end{aligned}$$

as coalgebras.

The results of the preceding calculations are summarized in the following theorem which determines the coalgebra $H_*\left(\Omega J_k\left(S^{2n}\right); \mathbb{Z}/p\mathbb{Z}\right)$ in a recursive sense.

Theorem 1.8. *If $k < p$, then $H_*\left(\Omega J_k\left(S^{2n}\right);\mathbb{Z}/p\mathbb{Z}\right) = \mathbb{S}^{\mathbb{Z}/p\mathbb{Z}}\langle y_1, z'_{k+1}\rangle$. For $k \geq p$, as coalgebras*

$$H_*\left(\Omega J_k\left(S^{2n}\right);\mathbb{Z}/p\mathbb{Z}\right) \cong \begin{cases} \bar{g}H_*\left(\mathcal{U}L'_1(k_1,np);\mathbb{Z}/p\mathbb{Z}\right)\coprod \mathbb{T}^{\mathbb{Z}/p\mathbb{Z}}\Big\langle \{{z'_p}^j\cdot z'_{k+1}\}_{j\geq 0}, \{{z'_p}^j\cdot c'_{(k_1+1)p}\}_{j\geq 0}, & \\ \{{z'_p}^j\cdot w_{(k_1+2)p}\}_{j\geq 0}, \{{z'_p}^j\cdot b_{k+p+1}\}_{j\geq 0}\Big\rangle \otimes \mathbb{S}^{\mathbb{Z}/p\mathbb{Z}}\langle y_1, z'_p, y_p\rangle, & \text{if } k\not\equiv -1(p); \\ \bar{g}H_*\left(\mathcal{U}L'_1(k_1,np);\mathbb{Z}/p\mathbb{Z}\right)\otimes \mathbb{S}^{\mathbb{Z}/p\mathbb{Z}}\langle y_1, z'_p, y_p\rangle, & \text{if } k\equiv -1(p). \end{cases}$$

where $H_*\left(\mathcal{U}L'_1(q,m);\mathbb{Z}/p\mathbb{Z}\right) = \mathbb{S}^{\mathbb{Z}/p\mathbb{Z}}\langle z'_{q+1}\rangle$ for $q < p$ and

$$\begin{aligned} &H_*\left(\mathcal{U}L'_1(q,m);\mathbb{Z}/p\mathbb{Z}\right) \cong H_*\left(\mathcal{U}L'_2(q,m);\mathbb{Z}/p\mathbb{Z}\right)\otimes \mathbb{S}^{\mathbb{Z}/p\mathbb{Z}}\langle z'_p, y_p\rangle \\ &\cong \begin{cases} \bar{g}H_*\left(\mathcal{U}L'_1(q_1,mp);\mathbb{Z}/p\mathbb{Z}\right) & \\ \qquad \coprod \mathbb{T}^{\mathbb{Z}/p\mathbb{Z}}\Big\langle \{{z'_p}^j\cdot z'_{q+1}\}_{j\geq 0}, \{{z'_p}^j\cdot c'_{(q_1+1)p}\}_{j\geq 0}, & \\ \{{z'_p}^j\cdot w_{(q_1+2)p}\}_{j\geq 0}, \{{z'_p}^j\cdot b_{q+p+1}\}_{j\geq 0}\Big\rangle \otimes \mathbb{S}^{\mathbb{Z}/p\mathbb{Z}}\langle z'_p, y_p\rangle, & \text{if } q\not\equiv -1(p); \\ \bar{g}H_*\left(\mathcal{U}L'_1(q_1,mp);\mathbb{Z}/p\mathbb{Z}\right)\otimes \mathbb{S}^{\mathbb{Z}/p\mathbb{Z}}\langle z'_p, y_p\rangle, & \text{if } q\equiv -1(p). \end{cases} \end{aligned}$$

for $q \geq p$. □

It is well known (and can be obtained as a limiting case of the preceding calculations) that

$$H_*\left(\Omega J\left(S^{2n}\right);\mathbb{Z}/p\mathbb{Z}\right) \cong \mathbb{S}^{\mathbb{Z}/p\mathbb{Z}}\langle v_{2n-1}, u_{2np-2}, v_{2np-1}, \ldots, u_{2np^t-2}, v_{2np^t-1}, \ldots\rangle$$

where $v_{2np^j-1} = \bar{g}^{j-1}y_{p,np^{j-1}}$ and $u_{2np^j-2} = \bar{g}^{j-1}z'_{p,np^{j-1}}$ for $j \geq 1$, with $v_{2n-1} = y_{1,n}$. Collecting our earlier results and repeatedly applying the preceding theorem gives the Hopf algebra structure of $H_*\left(\Omega J_k\left(S^{2n}\right);\mathbb{Z}/p\mathbb{Z}\right)$ described in the following theorem.

Theorem 1.9. *If $p^t \leq k < p^{t+1}$, then the Hopf algebra $H_*\left(\Omega J_k\left(S^{2n}\right);\mathbb{Z}/p\mathbb{Z}\right)$ is given by*

$$\begin{aligned} &H_*\left(\Omega J_k\left(S^{2n}\right);\mathbb{Z}/p\mathbb{Z}\right) \\ &\quad = \mathbb{T}^{\mathbb{Z}/p\mathbb{Z}}\Big\langle \{v_{2np^i-1}\}_{i=0}^t \cup \{u_{2np^i-2}\}_{i=1}^t\{z'_{k_t+1,np^t}\} \cup \{\bar{g}^t z'_{k_t+1,np^t}\} \\ &\qquad \cup \bigcup_{\substack{i=0 \\ k_i\not\equiv -1(p)}}^{t-1} \Big(\{u^j_{2np^{i+1}-2}\cdot \bar{g}^i z'_{k_i+1,np^i}\}_{j\geq 0} \cup \{u^j_{2np^{i+1}-2}\cdot \bar{g}^i c'_{(k_{i+1}+1)p,np^i}\}_{j\geq 0} \\ &\qquad \cup \{u^j_{2np^{i+1}-2}\cdot \bar{g}^i w_{(k_{i+1}+2)p,np^i}\}_{j\geq 0} \cup \{u^j_{2np^{i+1}-2}\cdot \bar{g}^i b_{k_i+p+1,np^i}\}_{j\geq 0}\Big)\Big\rangle \end{aligned}$$

modulo the relations

$$[u_{2np^i-2}, u_{2np^{i'}-2}] = 0, \qquad 1 \le i < i' \le t, \tag{1}$$

$$[u_{2np^i-2}, v_{2np^{i'}-1}] = 0, \qquad 1 \le i < i' \le t, \tag{2}$$

$$[v_{2np^i-1}, v_{2np^{i'}-1}] = 0, \qquad 1 \le i < i' \le t, \tag{3}$$

$$[v_{2np^{i+1}-1}, v_{2np^{i+1}-1}] = \begin{cases} \dfrac{\bar{g}^{i+1} z'_{k_{i+1}+1, np^{i+1}}}{u(2p)}, & \textit{if } k_{i+1} = 2; \\ 0, & \textit{otherwise}, \end{cases} \tag{4}$$

$$[v_{2np^{i+1}-1}, \bar{g}^{i'} z'_{k_{i'}+1, np^{i'}}] = 0, \qquad \begin{array}{l} 0 \le i < i' < t, k_{i'} \not\equiv -1(p) \ \textit{or} \\ 0 \le i < i' = t \end{array} \tag{5}$$

$$[v_{2np^{i+1}-1}, \bar{g}^{i} z'_{k_i+1, np^{i}}] = \bar{g}^i b_{k_i+p+1, np^i}, \qquad 0 \le i < t, k_i \not\equiv -1(p), \tag{6}$$

$$[v_{2np^{i+1}-1}, \bar{g}^{i'} c'_{(k_{i'+1}+1)p, np^{i'}}] = 0, \qquad 0 \le i < i' < t, k_{i'} \not\equiv -1(p), \tag{7}$$

$$[v_{2np^{i+1}-1}, \bar{g}^{i} c'_{(k_{i+1}+1)p, np^{i}}] = \bar{g}^i w_{(k_{i+1}+2)p, np^i}, \qquad 0 \le i < t, k_i \not\equiv -1(p), \tag{8}$$

$$[u_{2np^{i+1}-2}, \bar{g}^{i'} z'_{k_{i'}+1, np^{i'}}] = 0, \qquad 0 \le i < i'-1 < t, k_{i'} \not\equiv -1(p), \tag{9}$$

$$[u_{2np^{i+1}-2}, \bar{g}^{i+1} z'_{k_{i+1}+1, np^{i+1}}] = \bar{g}^i w_{(k_{i+1}+2)p, np^i}, \qquad 0 \le i < t, k_{i+1} \not\equiv -1(p), \tag{10}$$

$$[u_{2np^{i+1}-2}, \bar{g}^{i'} c'_{(k_{i'+1}+1)p, np^{i'}}] = 0, \qquad 0 \le i' < i < t, k_{i'} \not\equiv -1(p), \tag{11}$$

$$[v_{2n-1}, x] = \begin{cases} z'_{k+1}, & \textit{if } k = p^t \textit{ and } x = v_{2np^t-1}; \\ 0, & \textit{if } k \ne p^t \textit{ or } x \textit{ one of the other} \\ & \qquad\qquad \textit{generators}. \end{cases} \tag{12}$$

Proof. As a coalgebra this follows from repeated application of Theorem 1.8. Turning now to the algebra structure, we must reconstruct the multiplication in the extension $L_2 \to L_1 \to \mathbb{L}_{ab}\langle z_p, y_p\rangle$ and the extension $L_1 \to L_0 \to \mathbb{L}_{ab}\langle y_1\rangle$. By Lemma 1.3 we have the relations $[z'_p, x] = 0$ and $[y_p, x] = 0$ for

$$x \in \mathrm{Im}\Big(H_*\left(\mathcal{U}K'_2(k,n); \mathbb{Z}/p\mathbb{Z}\right) \cong \bar{g} H_*\left(\mathcal{U}K'_1(k_1, np); \mathbb{Z}/p\mathbb{Z}\right) \\ \to \bar{g} H_*\left(\mathcal{U}L'_1(k_1, np); \mathbb{Z}/p\mathbb{Z}\right)\Big).$$

This accounts for relations (1), (2), (3), (5), (7), (9), and (11). Lemma 1.7 gives (8), while (6) and (10) come from the definitions of b_{k_i+p+1} and $w_{(k_{i+1}+2)}$. Relation (4) comes from the fact that $[y_p, y_p] = a_{2p}$ belongs to I_2 except when $2p = (k_i+1)p$, in which case $a_{2p} = \bar{g} z_2^{\#}/u(2p)$ (after reduction modulo p). Finally the relation $[y_1, z'_m] = (1/2)[a_2, y_{m-1}]$ together with the definition shows $[y_1, x] = 0$ for any homology class x except when $k = p^t$, in which case $[y_1, y_{p^t}] = z'_{p^t+1}$. □

Corollary 1.10. *$J_k\left(S^{2n}\right)$ is mod p elliptic if and only if $k = qp^t - 1$ for some q and t.* □

For ease of notation, write s_j for $u\big((k_j+1)p\big)$. Our calculations also give the Bockstein action described in the following theorem.

Theorem 1.11. *Suppose $p^t \le k < p^{t+1}$.*

$$\beta(v_{2np^j-1}) = u_{2np^j-2}.$$

For i such that $k_i \not\equiv -1(p)$

$$\beta(\bar{g}^i w_{(k_{i+1}+2)p,np^i}) = u_{2np^{i+1}-2} \cdot \bar{g}^i c'_{(k_{i+1}+1)p,np^i}.$$

For i such that $i = t$ or $k_i \not\equiv -1(p)$

$$\begin{aligned} \beta^{(j)}(\bar{g}^i z'_{k_i+1,np^i}) &= 0 \qquad \text{if } j < i-q \\ \beta^{(i-q)}(\bar{g}^i z'_{k_i+1,np^i}) &= (-1)^{i-q-1} s_i s_{i-1} \dots s_{q+1} \bar{g}^{i-q} c'_{k_{q+1}+1,np^q} \end{aligned}$$

where $q = \max\{\sigma \mid \sigma < i$ and $k_\sigma \not\equiv -1(p)\}$, and

$$\beta^{(j)}(\bar{g}^i z'_{k_i+1,np^i}) = 0 \qquad \text{for all } j$$

if there does not exist $\sigma < i$ such that $k_\sigma \not\equiv -1(p)$.

For i such that $k_i \not\equiv -1(p)$

$$\beta(\bar{g}^i b_{k_i+p+1,np^i}) = u_{2np^i-2} \cdot \bar{g}^i z'_{k_i+1,np^i} - v_{2np^{i+1}-1} \cdot \beta(\bar{g}^i z'_{k_i+1,np^i}).$$

Proof. Since $d\bar{g} = \bar{g}d$ on K_1, the formulas for $\beta(v_{2np^j-1})$ and $\beta(\bar{g}^i w_{(k_{i+1}+2)p,np^i})$ are immediate. However d does not commute with $\bar{g}$ (until after reduction modulo p) on z'_{k_i+1} and b_{k_i+p+1}, so we must do the calculation directly on those elements. The definition, (9), gives for $0 \le j < i-q$,

$$\begin{aligned} d\bar{g}^j z'_{k_i+1,np^i} &= s_i d\bar{g}^{j-1} a'_{(k_i+1)p,np^i} - p s_i d\bar{g}^{j-1} z'_{k_{i-1}+1,np^{i-1}} \\ &= s_i d\bar{g}^{j-1} a'_{(k_i+1)p,np^{i-1}} - p s_i s_{i-1} d\bar{g}^{j-2} a'_{(k_{i-1}+1)p,np^{i-2}} \\ &\qquad + p^2 s_i s_{i-1} d\bar{g}^{j-2} z'_{k_{i-2}+1,np^{i-2}} \\ &= \dots \\ &= \sum_{t=0}^{j-1} (-1)^t p^t s_i s_{i-1} \cdots s_{i-t} d\bar{g}^{j-1-t} a'_{(k_{i-t}+1)p,np^{i-t-1}} \\ &\qquad + (-1)^j p^j s_i s_{i-1} \cdots s_{i-j+1} z'_{k_{i-j},np^{i-j}} \end{aligned}$$

and

$$d\bar{g}^{i-q} z'_{k_i+1,np^i} = \sum_{t=0}^{i-q-1} (-1)^t p^t s_i s_{i-1} \cdots s_{i-t} d\bar{g}^{i-q-1-t} a'_{(k_{i-t}+1)p,np^{i-t-1}}.$$

Since

$$\begin{aligned} d\bar{g}^{j-1-t} a'_{(k_{i-t}+1)p,np^{i-t-1}} &= \bar{g}^{j-1-t} d a'_{(k_{i-t}+1)p,np^{i-t-1}} \\ &= \bar{g}^{j-1-t} \left(p c'_{(k_{i-t}+1)p,np^{i-t-1}} + \frac{\bar{g} d z'_{k_{i-t}+1,np^{i-t}}}{s_{i-t}} \right) \end{aligned}$$

by (8), and

$$dz'_{k_m+1,np^m} = \begin{cases} c'_{(k_{m+1}+1)p,np^m}, & \text{if } i < m < q; \\ 0, & \text{if } m = i \text{ or } m = q, \end{cases}$$

the sum telescopes to give

$$d\bar{g}^j z'_{k_i+1,np^i} = \begin{cases} 0, & \text{if } j < i-q; \\ (-1)^{i-q-1} s_i s_{i-1} \dots s_{q+1} p^{i-q} c'_{(k_{q+1}+1)p,np^q}, & \text{if } j = i-q. \end{cases}$$

The final formula follows from the definition of b_{k_i+p+1,np^i} as $[y_{p,np^i}, z'_{k_i+1,np^i}]$. $\square$

Notice that the preceding theorem shows that there are some cases in which there is higher p-torsion in $H_*\left(\Omega J\left(S^{2n}\right)\right)$, something which does not happen in the limiting case $H_*\left(\Omega J\left(S^{2n}\right)\right)$. Since $\mathcal{P}^1_* u_{2np^j-2} = -u^p_{2np^{j-1}-2}$ in $H_*\left(\Omega J\left(S^{2n}\right); \mathbb{Z}/p\mathbb{Z}\right)$, [see CML], this relation also holds in $H_*\left(\Omega J_k\left(S^{2n}\right); \mathbb{Z}/p\mathbb{Z}\right)$ for $p^j \le k$. The smallest i such that $\bar{g}^i z'_{k_i+1,np^i}$ represents a homology class (i.e. $\max\{j \mid k_\sigma \equiv -1(p)$ for all $\sigma < j\}$) corresponds to the generator of $H_{2n(k+1)-2}\left(\Omega J_k\left(S^{2n}\right); \mathbb{Q}\right)$.

2. Product decompositions and exponents

In this section we will produce product decompositions of $\Omega J_k\left(S^{2n}\right)$ for $k < p^2 - p$ and use it to obtain homotopy exponents for these spaces. The first appearance of a non-trivial Steenrod operation $\mathcal{P}^1_*$ is in $\Omega J_{p^2-p}\left(S^{2n}\right)$ where $\mathcal{P}^1_*\left(\bar{g}z'_{p,np}\right) = -(u_{2np-2})^p$. The techniques of this paper are not sufficient to handle that situation. For $p \le k < p^2 - p$ we shall obtain a decomposition of the form

$$\Omega J_k\left(S^{2n}\right) \approx F_2(n) \times \Omega S^{2n(k+1)} \times \Omega \bigvee (\text{Moore spaces})$$

where $F_2(n)$ is a space whose homology is $\mathbb{S}^{\mathbb{Z}/p\mathbb{Z}}\langle v_{2n-1}, u_{2np-2}, v_{2np-1}\rangle$ which was introduced in [S2]. Because of the non-trivial Steenrod operations it is clear that a decomposition of $\Omega J_k\left(S^{2n}\right)$ must involve other types of spaces when $k \ge p^2 - p$.

It is well known and easy to check that $\Omega J_k\left(S^{2n}\right) \equiv S^{2n-1} \times \Omega S^{2n(k+1)-1}$ for $k < p-1$, and $\Omega J_{p-1}\left(S^{2n}\right)$ is easily seen to be atomic, although it is known to have an exponent from Toda's fibrations. We shall therefore concentrate on the cases $p \le k < p^2 - p$. Throughout this section, all homology will be assumed to be with $\mathbb{Z}/p\mathbb{Z}$ coefficients unless stated otherwise.

From section 1 we have

$$H_*\left(\Omega J_k\left(S^{2n}\right)\right) = \mathbb{S}^{\mathbb{Z}/p\mathbb{Z}}\langle v_{2n-1}, u_{2np-2}, v_{2np-1}\rangle \otimes \mathbb{T}^{\mathbb{Z}/p\mathbb{Z}}\langle B\rangle$$

where

$$B = \begin{cases} \{u^j_{2np-2} \cdot z'_{k+1}\}_{j\ge 0} \cup \{u^j_{2np-2} \cdot c'_{(k_1+1)p}\}_{j\ge 0} & \\ \qquad \cup \{u^j_{2np-2} \cdot w_{(k_1+2)p}\}_{j\ge 0} \cup \{u^j_{2np-2} \cdot b_{k+p+1}\}_{j\ge 0} \cup \{\bar{g}z'_{k_1+1,np}\}, & \\ & \text{if } k \not\equiv -1(p) \\ \{\bar{g}z'_{k_1+1,np}\}, & \text{if } k \equiv -1(p). \end{cases}$$

Since $k < p^2 - p$ it follows that $k_1 < p-1$, so the γ in the definition of $w_{(k_1+2)p}$ is 0 and thus $w_{(k_1+2)p} = u_{2np-2} \cdot a'_{(k_1+1)p}$. If $k \not\equiv -1(p)$, then $\bar{g}z'_{k_1+1,np} = a'_{(k_1+1)p}$, while if $k \equiv -1(p)$, then $\bar{g}z'_{k_1+1,np} = u\big((k_1+1)p\big)a'_{(k_1+1)p} - pz'_{k+1}$. Thus

$$B = \begin{cases} \{u^j_{2np-2} \cdot z'_{k+1}\}_{j\ge 0} \cup \{u^j_{2np-2} \cdot c'_{(k_1+1)p}\}_{j\ge 0} & \\ \qquad \cup \{u^j_{2np-2} \cdot a'_{(k_1+1)p}\}_{j\ge 0} \cup \{u^j_{2np-2} \cdot b_{k+p+1}\}_{j\ge 0}, & \text{if } k \not\equiv -1(p); \\ \tilde{a}_{k_1+1}, & \text{if } k \equiv -1(p) \end{cases}$$

where we have used $\tilde{a}_{k_1+1}$ to denote $u\big((k_1+1)p\big)a'_{(k_1+1)p} - pz'_{k+1}$.

Our immediate goal is to recall the definition of $F_2(n)$, obtain additional properties of it and related spaces, and show that it is a retract of $\Omega J_k\left(S^{2n}\right)$ for $p \le k < p^2 - p$.

Lemma 2.1. *For each k, there exists a map $T_k : \Omega J_k\left(S^{2n}\right) \to \Omega S^{2n(k+1)-1}$, which is onto on homology.*

Proof. This is well known, and there are several possible constructions [Gr], [MN]. One construction, following Toda [T], makes use of the co-action map $J_k\left(S^{2n}\right) \to J_k\left(S^{2n}\right) \vee S^{2nk}$ as follows. In general, $\Omega(X \vee Y) \approx \Omega X \times \Omega Y \times \Omega(\Omega X * \Omega Y)$. Define T_k to be the composite,

$$\begin{aligned}
&\Omega J_k\left(S^{2n}\right) \to \Omega\left(J_k\left(S^{2n}\right) \vee S^{2nk}\right) \xrightarrow{\pi_3} \Omega\left(\Omega J_k\left(S^{2n}\right) * \Omega S^{2nk}\right)\\
&\quad \approx \Omega\left(\Sigma \Omega J_k\left(S^{2n}\right) \wedge \Omega S^{2nk}\right) \to \Omega\left(\Omega J_k\left(S^{2n}\right) \wedge S^{2nk}\right) \to \Omega \Sigma^{2nk-1} J_k\left(S^{2n}\right)\\
&\quad \approx \Omega\left(\bigvee_{j=1}^{k} S^{2nk-1+2nj}\right) \to \Omega S^{2nk-1+2n} = \Omega S^{2n(k+1)-1}. \qquad \square
\end{aligned}$$

We refer to such a map as a Toda-Hopf invariant map.

As in [S2], we write K for the $2np-2$-skeleton of $\Omega J_{p-1}\left(S^{2n}\right)$. The following properties of K are shown in [S2].

Lemma 2.2.
1) $\Sigma^2 K \approx S^{2n+1} \vee S^{2np}$.
2) $\Sigma K \wedge K \approx S^{4n-1} \vee S^{2np+2n-2} \vee S^{2np+2n-2} \vee S^{4np-3}$. $\square$

For $0 \le k \le p$ let $X_k(n)$ denote the homotopy-fibre of the inclusion $J_k\left(S^{2n}\right) \hookrightarrow J_{p-1}\left(S^{2n}\right)$. From the Serre or Eilenberg-Moore spectral sequence we get

Proposition 2.3.

$$H^*\left(X_k(n)\right) = \begin{cases} \Lambda(\tilde{b}_{2n(k+1)-1}) \otimes \Gamma(\tilde{u}_{2np-2}), & \text{if } k < p-1;\\ 0, & \text{if } k = p-1,\end{cases}$$

where $\Lambda(\tilde{b})$ and $\Gamma(\tilde{u})$ denote respectively the exterior algebra on $\tilde{b}$ and the divided polynomial algebra on $\tilde{u}$. $\square$

Theorem 2.4. *For $1 \le k < p-1$, $\Sigma X_k(n) \approx \bigvee_{q=1}^{\infty} S^{q(2np-2)+1} \vee \bigvee_{q=0}^{\infty} S^{q(2np-2)+2n(k+1)}$.*

Proof. The theorem is equivalent to the statement that the Hurewicz homomorphism $h : \pi_*\left(\Sigma X_k(n)\right) \to H_*\left(\Sigma X_k(n)\right)$ is surjective. For $\epsilon = 0$ or 1, and $q \ge 0$, let $g_{q,\epsilon}$ denote the composite

$$K^q \times \left(S^{2n(k+1)-1}\right)^{\epsilon} \to \Omega J_{p-1}\left(S^{2n}\right) \times X_k(n) \xrightarrow{\mu} X_k(n),$$

where μ is the action of the fibre on the total space in the principal fibration induced from the fibration defining $X_k(n)$. The commutative diagram

$$\begin{array}{ccc}
\Omega J_{p-1}\left(S^{2n}\right) \times \Omega J_{p-1}\left(S^{2n}\right) & \longrightarrow & \Omega J_{p-1}\left(S^{2n}\right)\\
\downarrow & & \downarrow\\
\Omega J_{p-1}\left(S^{2n}\right) \times X_k(n) & \xrightarrow{\mu} & X_k(n)
\end{array}$$

shows that $g_{q,0_*}$ is an isomorphism on $H_{q(2np-2)}(\ )$. The commutativity of μ_* with the coproduct then shows that $g_{q,1_*}$ is an isomorphism on $H_{q(2np-2)+2n(k+1)-1}(\ )$. Since $\Sigma(X\times Y)\cong \Sigma X\vee\Sigma Y\vee\Sigma X\wedge Y$, the suspensions of the $g_{k,\epsilon}$'s can be used in conjuction with the splittings of Lemma 2.2 to produce maps from spheres exhibiting each element of $H_*\left(\Sigma X_k(n)\right)$ above degree $2np-1$ as an image under the Hurewicz homomorphism. Write B/A for the homotopy-theoretic cofibre of $A\to B$. Let $j: S^{2np-2}\to \Omega J_{p-1}\left(S^{2n}\right)/S^{2n-1}$ be a generator of the least non-vanishing homotopy group of that space. Let $c:\Omega J_{p-1}\left(S^{2n}\right)/\Omega J_k\left(S^{2n}\right)\to X_k(n)$ be induced from the fibration sequence $\Omega J_k\left(S^{2n}\right)\to\Omega J_{p-1}\left(S^{2n}\right)\to X_k(n)$. Then the composite

$$S^{2np-2}\xrightarrow{j}\Omega J_{p-1}\left(S^{2n}\right)/S^{2n-1}\to\Omega J_{p-1}\left(S^{2n}\right)/\Omega J_k\left(S^{2n}\right)\xrightarrow{c}X_k(n)$$

induces an isomorphism on $H_{2np-2}(\ )$, and so the the Hurewicz homomorphism is onto in this degree. Finally, the Hurewicz map is an isomorphism in the least non-vanishing degree, $2n(k+1)-1$, of $X_k(n)$. $\square$

Remark. In the case $X_0(n)=\Omega J_{p-1}\left(S^{2n}\right)$ the map c does not exist and the above argument gives only the well-known $\Sigma\Omega J_{p-1}\left(S^{2n}\right)\approx\Sigma K\vee\bigvee_{q=2}^{\infty}S^{q(2np-2)+1}\vee\bigvee_{q=1}^{\infty}S^{q(2np-2)+2np}$ as in [S2].

Corollary 2.5. *For $1\le k\le p-1$, $X_k(n)\approx S^{2n(k+1)-1}\times\Omega S^{2np-1}$.*

Proof. Using adjoints of maps coming from the preceding wedge decomposition, we get maps $X_k(n)\to\Omega S^{2np-1}$ and $X_k\to\Omega S^{2n(k+1)}\approx S^{2n(k+1)-1}\times\Omega S^{4n(k+1)-1}\to S^{2n(k+1)-1}$ which are onto on homology. Together they give a map $X_k(n)\to S^{2n(k+1)-1}\times\Omega S^{2np-1}$ which induces a homology isomorphism and thus is a homotopy equivalence. $\square$

For degree reasons, the restriction to $J_{qp^t-1}\left(S^{2n}\right)$ of the p^t th Hopf invariant map $H: J(S^{2n})\to J(S^{2np^t})$ lands in $J_{q-1}(S^{2np^t})$, the $2np^t(q-1)$-skeleton of $J(S^{2np^t})$. Note that in the notation of section 1, if $k_0=qp-1$, then $k_1=q-1$. The map $H: J_{qp-1}\left(S^{2n}\right)\to J_{q-1}(S^{2np})$ induces the map $h: K_2(k_0,n)\to K_1(k_1,np)$ of section 1.

Theorem 2.6. $J_{p^t-1}\left(S^{2n}\right)\hookrightarrow J_{qp^t-1}\left(S^{2n}\right)\xrightarrow{H}J_{q-1}(S^{2np^t})$ *is a fibration sequence up to homotopy.*

Proof. The proof is the same as that of the existence of Toda's homotopy-fibration $J_{p-1}\left(S^{2n}\right)\to J\left(S^{2n}\right)\to J\left(S^{2np}\right)$. Toda's calculation [T] shows that H induces an isomorphism on $\mathbb{Z}_{(p)}$-cohomology in degrees $2njp^t$ and consequently exhibits $H^*\left(J_{qp^t-1}\left(S^{2n}\right);\mathbb{Z}_{(p)}\right)$ as a free module over $H^*\left(J_{q-1}(S^{2np^t});\mathbb{Z}_{(p)}\right)$ with a basis for $H^*\left(J_{p^t-1}\left(S^{2n}\right);\mathbb{Z}_{(p)}\right)$ as basis. The Serre or Eilenberg-Moore spectral sequence thus shows that the homology of the homotopy-fibre of H is isomorphic to that of $J_{p^t-1}\left(S^{2n}\right)$. The composite $J_{p^t-1}\left(S^{2n}\right)\to J_{qp^t-1}\left(S^{2n}\right)\to J_{q-1}(S^{2np^t})$ is trivial for degree reasons and so yields a map from $J_{p^t-1}\left(S^{2n}\right)$ to the homotopy-fibre of H which induces the homology isomorphism above. $\square$

These Hopf invariant maps are compatible in the sense that if $q \leq q'$, then

$$\begin{array}{ccc} J_{qp^t-1}\left(S^{2n}\right) & \longrightarrow & J_{q'p^t-1}\left(S^{2n}\right) \\ \big\downarrow H & & \big\downarrow H \\ J_{q-1}(S^{2np^t}) & \longrightarrow & J_{q'-1}(S^{2np^t}) \end{array}$$

commutes, and the preceding theorem shows that this square is a homotopy-theoretic pullback. In particular, the homotopy-fibre of $J_{qp-1}\left(S^{2n}\right) \to J_{p^2-1}\left(S^{2n}\right)$ is $X_{q-1}(np)$, the same as that of $J_{q-1}(S^{2np}) \to J_{p-1}(S^{2np})$.

For $1 \leq q < p$, let j_q denote the composite $S^{2npq-1} \to X_{q-1}(np) \to J_{qp-1}\left(S^{2n}\right)$. In particular, j_1 is the composite $S^{2np-1} \to \Omega J_{p-1} S^{2np} \xrightarrow{\partial} J_{p-1}\left(S^{2n}\right)$ where the second map is induced from the fibration $J_{p-1}\left(S^{2n}\right) \to J_{p^2-1}\left(S^{2n}\right) \to J_{p-1}(S^{2np})$.

Lemma 2.7.

1) *For $1 \leq q < p$, j_q is the attaching map by means of which $J_{qp}\left(S^{2n}\right)$ is constructed from $J_{qp-1}\left(S^{2n}\right)$. In other words, $S^{2npq-1} \xrightarrow{j_q} J_{qp-1}\left(S^{2n}\right) \to J_{qp}\left(S^{2n}\right)$ is a cofibration sequence up to homotopy.*

2) *For $2 \leq q < p$, Ωj_q induces an injection on $H_*(\ )$.*

Proof. 1) Given a fibration $F \to E \to B$, Ganea [Ga] shows that the homotopy-fibre of the induced map $E/F \to B$ is $F * \Omega B$. The homotopy fibration $X_{q-1}(np) \to J_{qp-1}\left(S^{2n}\right) \to J_{p^2-1}\left(S^{2n}\right)$ thus yields a homotopy fibration

$$X_{q-1}(np) * \Omega J_{p^2-1}\left(S^{2n}\right) \to J_{qp-1}\left(S^{2n}\right)/X_{q-1}(np) \to J_{p^2-1}\left(S^{2n}\right)$$

which shows that $J_{qp-1}\left(S^{2n}\right)/X_{q-1}(np) \to J_{p^2-1}\left(S^{2n}\right)$ is more than $2npq$-connected. Since $S^{2npq-1} \to X_{q-1}(np)$ is also more than $2npq$-connected, the composite

$$J_{qp-1}\left(S^{2n}\right)/S^{2npq-1} \to J_{qp-1}\left(S^{2n}\right)/X_{q-1}(np) \to J_{p^2-1}\left(S^{2n}\right)$$

is $2npq$-connected, and so the homotopy cofibre of j_q is $J_{pq}\left(S^{2n}\right)$, the $2npq$-skeleton of $J_{p^2-1}\left(S^{2n}\right)$.

2) Looping before applying Ganea shows by the above argument that the composite

$$\Omega J_{qp-1}\left(S^{2n}\right)/\Omega S^{2npq-1} \to \Omega J_{qp-1}\left(S^{2n}\right)/\Omega X_{q-1}(np) \to \Omega J_{p^2-1}\left(S^{2n}\right)$$

is $2npq$-connected. Since $q > 1$, we know

$$H_*\left(\Omega J_{qp-1}\left(S^{2n}\right)\right) = \mathbb{S}^{\mathbb{Z}/p\mathbb{Z}}\langle v_{2n-1}, u_{2np-2}, v_{2np-1}\rangle \otimes \mathbb{T}^{\mathbb{Z}/p\mathbb{Z}}\langle \tilde{a}_{qp}\rangle,$$

where $|\tilde{a}_{qp}| = 2npq - 2$. However

$$H_*\left(\Omega J_{p^2-1}\left(S^{2n}\right)\right) = \mathbb{S}^{\mathbb{Z}/p\mathbb{Z}}\langle v_{2n-1}, u_{2np-2}, v_{2np-1}, u_{2np^2-2}\rangle$$

so this map can be $2npq$ connected only if Ωj_{q_*} is an injection in degree $2npq - 2$. Since it is multiplicative, this means that it is an injection in all degrees. $\square$

The space $F_2(n)$ was defined in [S2] as the homotopy-theoretic pullback of the maps $\Omega H : \Omega J(S^{2n}) \to \Omega J(S^{2np})$ and $E^2 : S^{2np-1} \to \Omega^2 S^{2np+1} \approx \Omega J(S^{2np})$. Since each of these maps is an H-map, this description makes it clear that $F_2(n)$ is an

H-space. The diagram

$$\begin{array}{ccccc} F_2(n) & \longrightarrow & \Omega J_{p^2-1}\left(S^{2n}\right) & \longrightarrow & \Omega J\left(S^{2n}\right) \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \Omega H} \\ S^{2np-1} & \longrightarrow & \Omega\left(J_{p-1}S^{2np}\right) & \longrightarrow & \Omega\left(JS^{2np}\right) \\ \downarrow & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} \\ J_{p-1}\left(S^{2n}\right) & = & J_{p-1}\left(S^{2n}\right) & = & J_{p-1}\left(S^{2n}\right) \end{array}$$

in which the top squares are homotopy-theoretic pullbacks and the columns are fibrations up to homotopy shows that $F_2(n)$ is also the homotopy fibre of the composite $S^{2np-1} \to \Omega\left(J_{p-1}S^{2np}\right) \xrightarrow{\partial} J_{p-1}\left(S^{2n}\right)$ which was earlier identified as j_1, the attaching map for the formation of $J_p\left(S^{2n}\right)$.

The diagram of homotopy fibrations

$$\begin{array}{ccc} F_2(n) & \overset{s_p}{\dashrightarrow} & \Omega J_p\left(S^{2n}\right) \\ \downarrow & & \downarrow \\ S^{2np-1} & \longrightarrow & PJ_p\left(S^{2n}\right) \\ \downarrow & & \downarrow \\ J_{p-1}\left(S^{2n}\right) & \longrightarrow & J_p\left(S^{2n}\right) \end{array}$$

produces an induced map s_p between the homotopy fibres. Although the homotopy class of s_p is not uniquely determined by the diagram (but depends on exactly how the homotopy commutative diagram is realized as a strictly commuting diagram of fibrations), any choice of s_p makes the diagram

$$\begin{array}{ccc} \Omega J_{p-1}\left(S^{2n}\right) & \longrightarrow & \Omega J_p\left(S^{2n}\right) \\ \downarrow{\scriptstyle \partial} & & \| \\ F_2(n) & \xrightarrow{s_p} & \Omega J_p\left(S^{2n}\right) \end{array}$$

homotopy commute and so induces an isomorphism on $H_{2n-1}(\ )$ and $H_{2np-2}(\ )$. For $k \geq p$ let s_k denote the composite $F_2(n) \to \Omega J_p\left(S^{2n}\right) \to \Omega J_k\left(S^{2n}\right)$. We will construct a left homotopy inverse to s_k for $p \leq k < p^2 - p$ by producing a left homotopy inverse to s_{p^2-p-1}.

Lemma 2.8.

1) *For $1 \leq q < p$ the homotopy fibre of $j_q : S^{2npq-1} \to J_{qp-1}\left(S^{2n}\right)$ is $F_2(n)$.*

2) *For $p \leq k < p^2 - p$, $s_k : F_2(n) \to \Omega J_k\left(S^{2n}\right)$ has a left homotopy inverse r_k.*

Proof. 1) We have already shown this for $q = 1$, so suppose $q > 1$. Let F be homotopy fibre of j_q. We know that $H_*\left(\Omega J_{qp-1}\left(S^{2n}\right)\right) = H_*\left(F_2(n)\right) \otimes H_*\left(\Omega S^{2npq-1}\right)$ and that Ωj_q induces an injection on homology. From this we can calculate that $H_*(F)$ equals $H_*\left(F_2(n)\right)$. The composite $F_2(n) \xrightarrow{s_{qp-1}} \Omega J_{qp-1}\left(S^{2n}\right) \xrightarrow{\partial} F$ induces an isomorphism on homology in degrees $2n-1$ and $2np-2$ and so is a homotopy equivalence by the "atomicity" style lemma which follows.

2) In part 1 this was shown for integers of the form $qp-1$ where $2 \leq q < p$, and so in particular for $p^2 - p - 1$. Therefore for all k the composite

$$\Omega J_k\left(S^{2n}\right) \to \Omega J_{p^2-p-1}\left(S^{2n}\right) \xrightarrow{r_{p^2-p-1}} F_2(n)$$

is a left homotopy inverse to s_k. □

Lemma 2.9. *Suppose that F and G are simply connected spaces such that $H_*(F) \cong H_*(G) \cong H_*\left(F_2(n)\right)$ as co-algebras. Let $f : F \to G$ be a map which induces an isomorphism on homology in degrees $2n-1$ and $2np-2$. Then f is a homotopy equivalence.*

Proof. Write a, u, and v for the algebra generators of $H_*\left(F_2(n)\right)$ in degrees $2n-1$, $2np-2$ and $2np-1$ respectively. Commutativity of f_* with the coproduct shows that the least degree in which f_* fails to be an isomorphism must contain a primitive. $PH_*\left(F_2(n)\right) = \left\langle a, v, u, u^p, \dots, u^{p^k}, \dots \right\rangle$. By hypothesis, $f_*(a) \neq 0$ and $f_*(u) \neq 0$. Commutativity with the Bockstein shows that $f_*(v) \neq 0$. Suppose by induction that $f_*(u^{p^m}) \neq 0$ for $m \leq k$. Then commutativity of f_* with the coproduct shows that $f_*(u^{p^k} v) \neq 0$ since there are no primitives in its degree. Applying β then shows that $f_*(u^{p^{k+1}}) \neq 0$ to complete the induction. Thus f_* is an injection on primitives and so is an isomorphism. Therefore f is a homotopy equivalence. $\square$

The final thing we need in order to produce our homotopy decomposition of $\Omega J_k\left(S^{2n}\right)$ for $p \leq k < p^2 - p$ is to show that certain elements in $H_*\left(\Omega J_k\left(S^{2n}\right)\right)$ are in the image of the mod p Hurewicz homomorphism.

Lemma 2.10. *Suppose $k \geq p$ and let u denote the generator of $H_*\left(\Omega J_k\left(S^{2n}\right)\right)$ in degree $2np-2$. If $x \in H_*\left(\Omega J_k\left(S^{2n}\right)\right)$ lies in the image of the mod p Hurewicz homomorphism, then so does $[u, x]$.*

Proof. Let $[\ ,\ ] : \Omega J_k\left(S^{2n}\right) \wedge \Omega J_k\left(S^{2n}\right) \to \Omega J_k\left(S^{2n}\right)$ be the Samelson product. Let i be the composite $K \to \Omega J_{p-1}\left(S^{2n}\right) \to \Omega J_k\left(S^{2n}\right)$. Suppose $x \in H_*\left(\Omega J_k\left(S^{2n}\right)\right)$ is the Hurewicz image of $f : P^N(p) \to J_k\left(S^{2n}\right)$. Then $[u, x]$ lies in the image of the map induced on homology by the composite $K \wedge P^N(p) \xrightarrow{i \wedge f} \Omega J_k\left(S^{2n}\right) \wedge \Omega J_k\left(S^{2n}\right) \xrightarrow{[\ ,\ ]} \Omega J_k\left(S^{2n}\right)$. However by Lemma 2.2, $K \wedge P^N(p) \cong P^{N+2n-1}(p) \vee P^{N+2np-2}(p)$, and so $[u, x]$ lies in the image of the mod p Hurewicz homomorphism. $\square$

Theorem 2.11. *For k such that $p \leq k < p^2 - p$ and $k \not\equiv -1(p)$, the elements z'_{k+1}, $a'_{(k_1+1)p}$, and b'_{k+p+1} of $H_*\left(\Omega J_k\left(S^{2n}\right)\right)$ lie in the image of the mod p Hurewicz homomorphism.*

Proof. Write simply z, a, and b for z'_{k+1}, $a'_{(k_1+1)p}$, and b'_{k+p+1} respectively. Let G be the homotopy fibre of $r_k : \Omega J_k\left(S^{2n}\right) \to F_2(n)$. z is the Hurewicz image of the composite $P^{2n(k+1)}(p) \to S^{2n(k+1)} \xrightarrow{j} G \to \Omega J_k\left(S^{2n}\right)$, where j is a generator of the least nonvanishing homotopy group of G. This composite will henceforth be written as τ. Let G' be the homotopy-theoretic fibre of the composite $G \to \Omega J_k\left(S^{2n}\right) \xrightarrow{T_k} \Omega S^{2n(k+1)-1}$. There is a map $P^{2npq-2}(p) \to G'$ which on homology hits the two least nonvanishing degrees of $H_*(G')$. The Hurewicz image of the composite $P^{2npq-2}(p) \to G' \to G \to \Omega J_p\left(S^{2n}\right)$ is a. Finally, writing L for the $2np-1$ skeleton of $F_2(n)$, we have from [S2] that $\Sigma^2 L \approx S^{2n+1} \vee P^{2np+1}(p)$. Since b lies in the image of the map induced on homology by the composite $P^{2n(k+1)} \wedge L \xrightarrow{\tau \wedge ()} \Omega J_k\left(S^{2n}\right) \wedge \Omega J_k\left(S^{2n}\right) \xrightarrow{[\ ,\ ]} \Omega J_k\left(S^{2n}\right)$, the wedge decomposition of $\Sigma^2 L$ shows that b lies in the image of the mod p Hurewicz homomorphism. $\square$

Similarly,

Theorem 2.12. *For k such that $p \leq k < p^2 - p$ and $k \equiv -1(p)$, the element $\tilde{a}_{k+1}$ lies in the image of the mod p Hurewicz homomorphism.'* □

Theorem 2.13. *For $k = qp + r$ such that $p \leq k < p^2 - p$ and $k \not\equiv -1(p)$*

$$\begin{aligned}\Omega J_k\left(S^{2n}\right) &\approx F_2(n)\\ &\times\Omega\left(S^{2n(k+1)-1}\vee\bigvee_{j=0}^{\infty}P^{2n(q+1)p+j(2np-2)-1}(p)\vee\bigvee_{j=0}^{\infty}P^{2n(k+p+1)+j(2np-2)-2}(p)\right).\end{aligned}$$

Proof. By Theorem 2.11 and Lemma 2.10, we can find maps $S^{2n(k+1)-2} \to \Omega J_k\left(S^{2n}\right)$, $P^{2n(q+1)p+j(2np-2)-1}(p) \to \Omega J_k\left(S^{2n}\right)$, and $P^{2n(k+p+1)+j(2np-2)-2}(p) \to \Omega J_k\left(S^{2n}\right)$ whose Hurewicz images are z, $ad^k(u)(a)$, and $ad^k(u)(b)$ respectively. Since $\Omega J_k\left(S^{2n}\right)$ is an H-space, these maps yield a map

$$\phi : \Omega\left(S^{2n(k+1)-1}\vee\bigvee_{j=0}^{\infty}P^{2n(q+1)p+j(2np-2)-1}(p)\vee\bigvee_{j=0}^{\infty}P^{2n(k+p+1)+j(2np-2)-2}(p)\right)\to\Omega J_k\left(S^{2n}\right)$$

by the universal property of the James construction. By construction the composite

$$F_2(n)\times\Omega\left(S^{2n(k+1)-1}\vee\bigvee_{j=0}^{\infty}P^{2n(q+1)p+j(2np-2)-1}(p)\vee\bigvee_{j=0}^{\infty}P^{2n(k+p+1)+j(2np-2)-2}(p)\right)\xrightarrow{s_k\times\phi}\Omega J_k\left(S^{2n}\right)\times\Omega J_k\left(S^{2n}\right)\to\Omega J_k\left(S^{2n}\right)$$

induces a homology isomorphism and is thus a homotopy equivalence. □

Similarly

Theorem 2.14. *For k such that $p \leq k < p^2 - p$ and $k \equiv -1(p)$*

$$\Omega J_k\left(S^{2n}\right) \approx F_2(n)\times\Omega S^{2n(k+1)-1}. \quad \square$$

Applying the Hilton-Milnor theorem and the main theorems of [CMN3] and [N3] gives

Corollary. *$J_k\left(S^{2n}\right)$ has a homotopy exponent for $k < p^2 - p$.* □

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